Section 15.1

Multivariable Functions

Thus far in this course, we have considered the calculus of functions $\vec{r}: \mathbb{R} \to \mathbb{R}^3$.

We now return to scalar valued functions $f : D \to \mathbb{R}$ with $D \subset \mathbb{R}^n$.

Example 15.1.1. Find the domain of $g(x, y) = \sqrt{4 - x^2 - y^2}$

Functions can be defined implicitly via the form $F(x, y, z) = 0$. Such an expression gives z as a function of x and y only if the graph of the relation satisfies the vertical line test.

Example 15.1.2. Find a function $f(x, y)$ whose graph is the lower half of the hyperboloid of two sheets, $-x^2 - y^2 + z^2 = 1$

We can understand multivariable functions through the same ways we examined quadric surfaces: **level curves** or $f(x, y) = z_0$ traces.

Example 15.1.3. Sketch the level curves of the surface $f(x, y) = y - x^2 - 1$.

Example 15.1.4. Sketch the level curves of the surface $f(x, y) = 2 + \sin(x - y)$.

Example 15.1.5. What are the level sets of the function $w = f(x, y, z)$ given by

$$f(x, y, z) = \sqrt{z - x^2 - y^2}.$$

Section 15.2

Limits

For a point xa on the real number line, points close to a satisfy $-\delta < x - a < \delta$. Thus to determine a limit of a function $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{x \to a} f(x)$$

we need only consider values of x slightly above and below a.

For points $P_0(a, b) \in \mathbb{R}^2$, there are more "sides" of P_0 to be close on, in particular we need $\sqrt{(x-a)^2 + (y-b)^2} < \delta$. That is, an entire disc of close points. This factors into the definition of the limit.

Definition: Limits
The function f has the **limit** L as $P(x, y)$ approaches $P_0(a, b)$, written

$$\lim_{(x,y) \to (a,b)} f(x, y) = \lim_{P \to P_0} f(x, y) = L,$$

if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x, y) - L| < \varepsilon$$

whenever (x, y) is in the domain of f and

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

From here, it is not far to the following theorem.

Theorem 15.1: Limits of Linear Functions
Let a, b and c be real numbers.

1. Constant function $f(x, y) = c$: $\lim\limits_{(x,y) \to (a,b)} c = c.$
2. Linear function $f(x, y) = x$: $\lim\limits_{(x,y) \to (a,b)} x = a.$
3. Linear function $f(x, y) = y$: $\lim\limits_{(x,y) \to (a,b)} y = b.$

To enhance our ability to compute limits, as with limits of vector valued functions, we have the following limit laws.

Theorem 15.2. Limit Laws of Functions of Two Variables
Let L and M be real numbers and suppose $\lim_{(x,y)\to(a,b)} f(x,y) = L$, $\lim_{(x,y)\to(a,b)} g(x,y) = M$. Assume c is a constant and $n > 0$ is an integer.

1. **Sum** $\lim_{(x,y)\to(a,b)} (f(x,y) + g(x,y)) = L + M$
2. **Difference** $\lim_{(x,y)\to(a,b)} (f(x,y) - g(x,y)) = L - M$
3. **Constant multiple** $\lim_{(x,y)\to(a,b)} cf(x,y) = cL$
4. **Product** $\lim_{(x,y)\to(a,b)} (f(x,y)g(x,y)) = LM$
5. **Quotient** $\lim_{(x,y)\to(a,b)} (f(x,y)/g(x,y)) = L/M$ provided $M \neq 0$
6. **Power** $\lim_{(x,y)\to(a,b)} f(x,y)^n = L^n$
7. **Root** $\lim_{(x,y)\to(a,b)} f(x,y)^{1/n} = L^{1/n}$

Example 15.2.6. Evaluate $\lim_{(x,y)\to(2,8)} (3x^2y + \sqrt{xy})$

Definition: Interior and Boundary Points
Let R be a region in $\mathbb{R}^2$. An **interior point** P of R lies entirely within R, that is, it is possible to find a disc centered at P that contains only points of R.

A **boundary point** Q of R lies on the edge of R in the sense that every disc centered at Q contains at least one point in R and one point not in R.

Definition: Open and Closed Sets
A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points.

For a point $P_0(a, b)$ of the boundary of the domain of f, the limit $\lim_{(x,y)\to(a,b)} f(x, y)$ can exist even if P_0 is not in the domain of f provided that $f(x, y)$ approaches the same value as (x, y) approaches (a, b) along all paths that lie in the domain of f.

Example 15.2.7. Evaluate $\lim_{(x,y)\to(4,1)} \dfrac{xy - 4y^2}{\sqrt{x} - 2\sqrt{y}}$

Example 15.2.8. Evaluate the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x+y)^2}{x^2+y^2}$

Definition: Continuity
The function f is continuous at the point (a,b) provided

1. f is defined at (a,b)
2. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exists, and
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

From Theorems 15.1 and 15.2 we get polynomials and rational functions are continuous on their domain.

Example 15.2.9. Determine at which points the following function is continuous

$$f(x,y) = \begin{cases} \dfrac{3xy^2}{x^2+y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Theorem 15.3: Continuity of Compositions
If $u = g(x, y)$ is continuous at (a, b) and $z = f(u)$ is continuous at $g(a, b)$, then the composite function $z = f(g(x, y))$ is continuous at (a, b).

Example 15.2.10. Determine at which points is the function $h(x, y) = \ln(x^2+y^2+4)$ continuous.